BIFURCATION OF PERIODIC SOLUTION FROM AN EQUILIBRIUM POINT IN THE MULTIPARAMETER CASE

H. Shariati

Department of Mathematics, Faculty of Sciences, University of Sistan and Baluchistan, Zahedan, Islamic Republic of Iran

Abstract

We consider the bifurcation of periodic solutions from an equilibrium point of the given equation: $\dot{x}=F(x,\varepsilon)$, where $x \in R^{n+m}$, ε is a vector of m real parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ and $F:R^{n+m} \times R^m \to R^{n+m}$ has at least second continuous derivations in variables.

Introduction

In this paper we consider the bifurcation of periodic solutions from an equilibium point. The term bifurcation of periodic solutions from an equilibrium point refers here simply to the behaviour of periodic solutions of the given equation

$$\dot{x} = F(x,\varepsilon), \tag{1.1}$$

for values of ε near O, where $x \in R^{n+m}$, ε is a vector of m small real parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ and $F: R^{n+m} \times R^m \to R^{n+m}$ has continuous second derivatives in all variables.

Suppose $x_o(t) = o$ is an equilibrium point of system (1.1) for all $\varepsilon \in R^m$. Thus one can expand $F(x, \varepsilon)$ about $x_o(t) = 0$ to obtain

$$\dot{x} = A(\varepsilon)x + f(x, \varepsilon), \tag{1.2}$$

where $A(\varepsilon)$ is a differentiable $(n+m) \times (n+m)$ matrix depending only on ε and $f(x, \varepsilon)$ the non-linear part of F is at least a twice differentiable function of (x, ε) such that

$$|f(x, \varepsilon)| = O(|x|),$$

uniformly in ε for $\|\varepsilon\|$ sufficiently small. ($\|\varepsilon\|$ is euclidean norm). We assume that the matrix $A(\varepsilon)$ has two conjugate complex eigenvalues

$$\alpha(\varepsilon) \pm i\,\omega(\varepsilon), \tag{1.3}$$

where $\alpha(\varepsilon)$ and $\omega(\varepsilon)$ are differentiable functions with respect to the components of ε and

$$\alpha(o) = 0, \tag{1.4}$$

$$\omega(o) \neq 0, \tag{1.5}$$

The remaining eigenvalues of A(o) have non-zero real parts.

By means of an appropriate transformation $x \to P(\varepsilon)x$ with a non-singular smooth matrix $P(\varepsilon)$ and by means of the time scaling $t \to \frac{t}{\omega(\varepsilon)}$ (1.2) may be written as

$$\dot{x} = \begin{bmatrix} J(\varepsilon) & o \\ o & D(\varepsilon) \end{bmatrix} x + g(x, \varepsilon) = B(\varepsilon)x + g(x, \varepsilon), \tag{1.6}$$

where $g(x, \varepsilon) = \frac{1}{\omega(\varepsilon)} p^{-1} f(Px, \varepsilon)$,

$$J(\varepsilon) = \begin{bmatrix} \lambda(\varepsilon) & 1 \\ -1 & \lambda(\varepsilon) \end{bmatrix}, B(\varepsilon) = \begin{bmatrix} J(\varepsilon) & o \\ o & D(\varepsilon) \end{bmatrix}, \tag{1.7}$$

such that $\lambda(O) = O$ and $D(\varepsilon)$ is an $(n+m-2) \times (n+m-2)$ matrix.

We discuss 2π-periodic solution of (1.6) to obtain

Keywords: Bifurcation; Periodic solution; Equilibrium point

$\frac{2\pi}{\omega(\varepsilon)}$ -periodic solution of (1.2). The non-trivial solution of (1.6) is denoted by $x(t, C, \varepsilon)$, where C is an (n+m)-vector such that [1]

$$x(t, C, \varepsilon), = C.$$

In Section 2, we give an account of a general analysis using the implicit function theorem [2] and in Section 2 we impose some hypothesis on the initial value C and transversality condition [1] so that the implicit function theorem can be applied to prove the existence of the uniques solution to (1.6). Also, we discuss the problem when the initial value has no restriction. In this way, we shall extend the Hopf bifurcation theorem from the single parameter to the multi-parameter case.

Section 2

We start with a real (n+m)-dimensional autonomous differential system (1.6) and discuss the existence of 2π-periodic solutions for that equation.

Theorem 2.1

Suppose that the matrix $B(\varepsilon)$ in (1.6) has a pair of complex eigenvalues

$$\begin{array}{l}\lambda(\varepsilon) + i,\\ \lambda(\varepsilon) - i,\end{array} \qquad (2.1)$$

where $\lambda(\varepsilon)$ is a differentiable function with respect to the components of ε and

$$\lambda(o) = o, \frac{d\lambda}{d\varepsilon_j}(o) \neq o \text{ for some } J=1,2,\dots m \qquad (2.2)$$

i is an eigenvalue of multiplicity one of B(O). The remaining n+m-2 eigenvalues of B(O) have non-zero real parts, in other words, the matrix B(O) has no eigenvalues of the form qi,

$$q = 0, \pm 2, \dots . \qquad (2.3)$$

Then there is a neighbourhood U of 0 in R^m and there exist real valued differentiable functions

$$C_1, C_2, \dots, C_{n+m},$$

with the following properties:

(a) C_i (3), i = 1, ..., n+m are deffined for all $\varepsilon \in U$.

(b) $C_i(O) = O$, i = 1, ..., n+m.

(c) For any $\varepsilon \in U$,

$$C_i(\varepsilon) = O(\|\varepsilon\|)$$

i = 1, ..., n+m, then there exist non-unique 2π-periodic solution $x(t, C(\varepsilon), \varepsilon)$ of (1.6), and consequently non-unique $\frac{2\pi}{\omega(\varepsilon)}$ -periodic solution of (1.2).

Proof

To meet the hypothesis (2.1)-(2.2) of the theorem we shall assume that

$$B(o) = \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & D(0) \end{bmatrix}, \qquad (2.5)$$

where D(0) is an (n+m-2) × (n+m-2) matrix with no eigenvalues of the form qi, $q = 0, \pm 1, \pm 2$.

As was mentioned before, $x(t, C, \varepsilon)$ is a general solution of (1.6). This solution has period 2π if and only if

$$x[2\pi, C, \varepsilon] - x[o, C, \varepsilon] = 0. \qquad (2.6)$$

Now we search for a vector $C \in R^{n+m}$ such that (2.6) is satisfied. For (1.6) we use the variation of constant formula [1] to obtain

$$x(t, C, \varepsilon) = e^{tB(\varepsilon)} C + \int_0^t e^{(t-\sigma)B(\varepsilon)} g[x(\sigma, C, \varepsilon), \varepsilon] d\sigma.$$

Thus

$$x(2\pi, C, \varepsilon) = e^{2\pi B(\varepsilon)} C + \int_0^{2\pi} e^{(2\pi-\sigma)B(\varepsilon)} g[x(\sigma, C, \varepsilon), \varepsilon] d\sigma.$$

and hence (2.6) may be written as

$$\left[e^{2\pi B(\varepsilon)} - I\right] C + \int_0^{2\pi} e^{(2\pi-\sigma)B(\varepsilon)} g[x(\sigma, C, \varepsilon), \varepsilon] d\sigma = 0. \qquad (2.7)$$

Since $g(x, \varepsilon)$ is twice continuously differentiable and

$$|g(x, \varepsilon)| = 0(|x|),$$

we have that

$$g(x, \varepsilon) = \frac{1}{2} D_x^2 g(0, 0) x^2 + 0(|x|^2).$$

On combining this with the representation for $x(t, C, \varepsilon)$ given by the variation of constant formula we will have

$$x(t, C, \varepsilon) = e^{tB(\varepsilon)} C + 0(|C|).$$

Therefore (2.7) is equivalent to

$$\left[e^{2\pi B(\varepsilon)} - I\right] C + 0(|C|) = 0. \qquad (2.8)$$

We define $H: R^{n \times m} \times R^m \to R^{n+m}$ by

$$H(C, \varepsilon) = \left[e^{2\pi B(\varepsilon)} - I\right] C + O(|C|), \qquad (2.9)$$

and solve $H(C, \varepsilon) = 0$ for $C_1, C_2, \dots, C_{n+m}$ in terms of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$. Clearly H (O, O) = O, thus $H(C, \varepsilon)$ is defined at the origin. Using Taylor's expansion on the right of (2.9) we have

$$H(C, \varepsilon) = \left|(2\pi \frac{\partial B}{\partial \varepsilon_1}(0) e^{2\pi B(o)} \varepsilon_1, \dots, + 2\pi \frac{\partial B}{\partial \varepsilon}(0) e_m) e^{2\pi B(o)} + e^{2\pi B(o)} - I\right| C +$$

$L(\varepsilon_1, \ldots, \varepsilon_m)\, C + O(|C|)$, (2.10)

where $L(\varepsilon_1, \ldots, \varepsilon_m)$ is an $(n+m) \times (n+m)$ matrix all of whose elements α_{ij} are at least second order in $\varepsilon_1, \ldots, \varepsilon_m$, i.e.

$$\lim_{\|\varepsilon\| \to 0} \frac{|a_{ij}(e_1, \ldots, \varepsilon_m)|}{|\varepsilon_2| + \ldots + |\varepsilon_m|} = 0.$$

Considering the matrix $B(\varepsilon)$ and its eigenvalues as mentioned in (2.1), we can write

$$B(\varepsilon) = \begin{bmatrix} \lambda(\varepsilon) & 1 & \\ -1 & \lambda(\varepsilon) & \\ & & D(\varepsilon) \end{bmatrix}.$$

Thus for all $j = 1, \ldots, m$ we have

$$\frac{\partial B}{\partial \varepsilon_j}(0) = \begin{bmatrix} \frac{\partial \lambda}{\partial \varepsilon_j}(0) & 0 & \\ 0 & \frac{\partial \lambda}{\partial \varepsilon_j}(0) & \\ & & \frac{\partial D}{\partial \varepsilon_j}(0) \end{bmatrix} \qquad (2.11)$$

From (2.11) and the fact that

$$e^{2\pi B(0)} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & e^{2\pi D(0)} \end{bmatrix},$$

(2.10) is written as

$$\left\{ 2\pi \begin{bmatrix} \frac{\partial \lambda}{\partial \varepsilon_1}(0) & 0 & \\ 0 & \frac{\partial \lambda}{\partial \varepsilon_1}(0) & \\ & & \frac{\partial D}{\partial \varepsilon_1}(0) \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & e^{2\pi D(0)} \end{bmatrix} \varepsilon_1 \right.$$

$$= 2\pi \begin{bmatrix} \frac{\partial \lambda}{\partial \varepsilon_2}(0) & 0 & \\ 0 & \frac{\partial \lambda}{\partial \varepsilon_2}(0) & \\ & & \frac{\partial D}{\partial \varepsilon_2}(0) \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & e^{2\pi D(0)} \end{bmatrix} \varepsilon_2 + \ldots + 2\pi \begin{bmatrix} \frac{\partial \lambda}{\partial \varepsilon_m}(0) & 0 & \\ 0 & \frac{\partial \lambda}{\partial \varepsilon_m}(0) & \\ & & \frac{\partial D}{\partial \varepsilon_m}(0) \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & e^{2\pi D(0)} \end{bmatrix} \varepsilon_m$$

$$\left. + \begin{bmatrix} 0 & 0 & \\ 0 & 0 & \\ & & e^{2\pi D(0)} - I \end{bmatrix} \right\} \begin{bmatrix} \frac{C_1}{\|\varepsilon\|} \\ \frac{C_2}{\|\varepsilon\|} \\ C_3 \\ \vdots \\ C_{n+m} \end{bmatrix} + L(\varepsilon_1, \ldots, \varepsilon_m) \begin{bmatrix} \frac{C_1}{\|\varepsilon\|} \\ \frac{C_2}{\|\varepsilon\|} \\ C_3 \\ \vdots \\ C_{n+m} \end{bmatrix} + 0(|C|)$$

$$= \left\{ 2\pi \begin{bmatrix} \frac{\partial \lambda}{\partial \varepsilon_1}(0) & 0 & \\ 0 & \frac{\partial \lambda}{\partial \varepsilon_1}(0) & \\ & & \frac{\partial D}{\partial \varepsilon_1}(0)\, \varepsilon^{2\pi D(0)} \end{bmatrix} \varepsilon_1 + \ldots + 2\pi \begin{bmatrix} \frac{\partial \lambda}{\partial \varepsilon_m}(0) & 0 & \\ 0 & \frac{\partial \lambda}{\partial \varepsilon_m}(0) & \\ & & \frac{\partial D}{\partial \varepsilon_m}(0)\, \varepsilon^{2\pi D(0)} \end{bmatrix} \varepsilon_m + \right.$$

$$\begin{bmatrix} 0 & 0 & \\ 0 & 0 & \\ & & e^{2\pi D(0)}-I \end{bmatrix} \left[\begin{matrix} \frac{C_1}{\|\varepsilon\|} \\ \frac{C_2}{\|\varepsilon\|} \\ C_3 \\ \vdots \\ C_{n+m} \end{matrix} \right] + L(\varepsilon_1, \dots, \varepsilon_m) \left[\begin{matrix} \frac{C_1}{\|\varepsilon\|} \\ \frac{C_2}{\|\varepsilon\|} \\ C_3 \\ \vdots \\ C_{n+m} \end{matrix} \right] + 0(|C|) =$$

$$\begin{bmatrix} 2\pi \sum_{\varphi=1}^{m} \frac{\partial \lambda}{\partial \varepsilon_j}(0)\, \varepsilon_j & 0 & \\ 0 & 2\pi \sum_{j=1}^{m} \frac{\partial \lambda}{\partial \varepsilon_j}(0)\, \varepsilon_j & \\ & & 2\pi e^{2\pi D(0)} \sum \frac{\partial D}{\partial \varepsilon_j}(0)\, \varepsilon_j + e^{2\pi D(0)} - I \end{bmatrix}$$

$$\left[\begin{matrix} \frac{C_1}{\|\varepsilon\|} \\ \frac{C_2}{\|\varepsilon\|} \\ C_3 \\ \vdots \\ C_{n+m} \end{matrix} \right] + L(\varepsilon_1, \dots, \varepsilon_m) \left[\begin{matrix} \frac{C_1}{\|\varepsilon\|} \\ \frac{C_2}{\|\varepsilon\|} \\ C_3 \\ \vdots \\ C_{n+m} \end{matrix} \right] + 0(|C|) =$$

$$\left[\begin{matrix} 2\pi \sum_{1}^{m} \frac{\partial \lambda}{\partial \varepsilon_j}(0) \frac{\varepsilon_j}{\|\varepsilon\|} \times C_1 \\ 2\pi \sum_{1}^{m} \frac{\partial \lambda}{\partial \varepsilon_j}(0) \frac{\varepsilon_j}{\|\varepsilon\|} \times C_2 \\ \left[2\pi e^{2\pi D(0)} \left(\sum \frac{\partial D}{\partial \varepsilon_j}(0)\, \varepsilon_j \right) + e^{2\pi D(0)} - I \right] \left[\begin{matrix} C_3 \\ C_4 \\ \vdots \\ C_{n+m} \end{matrix} \right] \end{matrix} \right] + L(\varepsilon_2, \dots, \varepsilon_m) \left[\begin{matrix} \frac{C_1}{\|\varepsilon\|} \\ \frac{C_2}{\|\varepsilon\|} \\ C_3 \\ \vdots \\ C_{n+m} \end{matrix} \right] + 0(|C|).$$

Suppose $\sum_{j=1}^{m} \frac{\partial \lambda}{\partial \varepsilon_j}(0)\, 1_j \neq 0$ where 1_j, $j = 1, \dots, m$ are direction cosine of vector ε when $\varepsilon \longrightarrow 0$ which forces $\frac{\partial \lambda}{\partial \varepsilon_j}(0) \neq 0$ for some $j = 1, \dots, m$. Then the functional determinant of the vector function $H(C, \varepsilon)$ evaluated at the origin is

$$\det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{2\pi D(0)} - I \end{bmatrix} = 0.$$

Therefore, by Theorem (1.5) of [2] there exists a neighbourhood U of 0 in $\mathbb{R}^m$ such that for any $\varepsilon \in U$ the equation (2.9) has non-unique solutions. We shall need in the sequel to examine the second and higher order terms. Thus, reconsider the equation

$H(C, \varepsilon) = (e^{2\pi B(\varepsilon)} - I)C + A(C),\ A(C) = 0(|C|),$

where

$$(e^{2\pi B(\varepsilon)} - I)C = \begin{bmatrix} 2\pi \sum_{j=1}^{m} \frac{\partial \lambda}{\partial \varepsilon_j}(0)\,\varepsilon_j C_1 \\ 2\pi \sum \frac{\partial \lambda}{\partial \varepsilon_j}(0)\,\varepsilon_j C_2 \\ (2\pi e^{2\pi D(0)} \sum \frac{\partial D}{\partial \varepsilon_j}(\varepsilon)\,\varepsilon_j + e^{2\pi D(0)} - I \begin{bmatrix} C_3 \\ \vdots \\ C_{n+m} \end{bmatrix} \end{bmatrix} + L(\varepsilon_1, \ldots, \varepsilon_m)$$

$$L(\varepsilon_1, \ldots, \varepsilon_m) = \frac{1}{2!}\left\{ \sum_{i,j}^{m} \left[2\pi \frac{\partial^2 B}{\partial \varepsilon_i \partial \varepsilon_j}(0) + 4\pi^2 \left(\frac{\partial B(0)}{\partial \varepsilon_i}\right)\left(\frac{\partial B(0)}{\partial \varepsilon_j}\right)\right] e^{2\pi B(0)}\,\varepsilon_i \varepsilon_j + 0(|\varepsilon_i \varepsilon_j|)\, C \right\} =$$

$$\begin{bmatrix} \left\{ \frac{1}{2!} \sum_{i,j}^{m} \left[2\pi \frac{\partial^2 \lambda}{\partial \varepsilon_i \varepsilon_j}(0) + 4\pi^2 \left(\frac{\partial \lambda(0)}{\partial \varepsilon_i}\right)\left(\frac{\partial \lambda(0)}{\partial \varepsilon_j}\right)\right] \varepsilon_i \varepsilon_j + 0(|\varepsilon_i \varepsilon_j|) \right\} C_1 \\ \left\{ \frac{1}{2!} \sum \left[2\pi \frac{\partial^2 \lambda}{\partial \varepsilon_i \partial \varepsilon_j}(0) + 4\pi^2 \left(\frac{\partial \lambda(0)}{\partial \varepsilon_i}\right)\left(\frac{\partial \lambda(0)}{\partial \varepsilon_j}\right)\right] \varepsilon_i \varepsilon_j + 0\,(|\varepsilon_i \varepsilon_j + 0(|\varepsilon_i \varepsilon_j|) \right\} C_2 \\ \frac{1}{2!} \sum \left[2\pi \frac{\partial^2 D}{\partial \varepsilon_i \partial \varepsilon_j}(0)\, e^{2\pi D(0)} + 4\pi^2 \left(\frac{\partial D(0)}{\partial \varepsilon_i}\right)\left(\frac{\partial D(0)}{\partial \varepsilon_j}\right) e^{2\pi D(0)}\right] \varepsilon_i \varepsilon_j + 0(|\varepsilon_i \varepsilon_j|) \begin{bmatrix} C_3 \\ \vdots \\ C_{n+m} \end{bmatrix} \end{bmatrix}$$

and

$$A(C) = \int_0^{2\pi} e^{(2\pi - \sigma) B(\varepsilon)} \times g(x(\sigma, C, \varepsilon), \varepsilon)\, d\sigma \ = \int_0^{2\pi} \Big\{ e^{(2\pi - \sigma) B(0)} + (2\pi - 6) \sum \frac{\partial B}{\partial \varepsilon_j}(0)\, \varepsilon^{(2\pi - \sigma) B(0)} \varepsilon_j +$$

$$\frac{1}{2!} \sum \left[(2\pi - \sigma) \frac{\sigma^2 B}{\partial \varepsilon_i \partial \varepsilon_j}(0) + 4\pi^2 \left(\frac{\partial B}{\partial \varepsilon_i}\right)\left(\frac{\partial B}{\partial \varepsilon_j}\right)\right] e^{(2\pi - \sigma) B(0)} \varepsilon_i \varepsilon_j +$$

$$0(|\varepsilon_i \varepsilon_j|)\Big\} \times \frac{1}{2} D_x^2 g(0,0)\, x^2 d\sigma + 0(|x^2|) =$$

$$\frac{1}{2} \int_0^{2\pi} \left\{ I + (2\pi - \sigma) \sum \frac{\partial B}{\partial \varepsilon_j}(0)\,\varepsilon_j + \frac{1}{2!} \sum \left[(2\pi - \sigma) \frac{\partial^2 B}{\partial \varepsilon_i \partial \varepsilon_j}(0) + (2\pi - \alpha)^2 \left(\frac{\partial B}{\partial \varepsilon_i}\right)\left(\frac{\partial B}{\partial \varepsilon_j}\right)\right] \varepsilon_i \varepsilon_j + 0(|\varepsilon_i \varepsilon_j|) \right\} \times$$

$$\begin{bmatrix} \cos\sigma & -\sin\sigma & \\ \sin\sigma & \sin\sigma & \\ & & e^{(2\pi - \sigma) D(0)} \end{bmatrix} \times \begin{bmatrix} \sum_{i,j} & \alpha_{i,j}^{(1)} & x_i x_j \\ & \vdots & \\ & \vdots & \\ \sum_{i,j} & \alpha_{ij}^{(n+m)} & x_i x_j \end{bmatrix} d\sigma$$

where

$$\alpha_{ij}^{(k)} = \frac{\partial^2 g(k)}{\partial x_i \partial x_j}(0,0),\ K= 1,2,\dots, n+m,$$

or

$$AC=\begin{bmatrix} \frac{1}{2}\int_0^{2\pi}\left\{1+(2\pi-\sigma)\sum_{j=1}^{m}\frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j+\frac{1}{2!}\sum\left[(2\pi-\sigma)\frac{\sigma^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0)+(2\pi-\sigma)^2\left(\frac{\sigma\lambda(0)}{\partial\varepsilon_i}\right)\left(\frac{\sigma\lambda(0)}{\partial\varepsilon_i}\right)\right]\right. \\ \left.\varepsilon_i\varepsilon_j+0\left(\left|\varepsilon_i\varepsilon_j\right|\right)\right\}\times\cos\sigma\sum_{i,j}\alpha_{ij}^{(1)}x_ix_j-\sin\sigma\sum_{i,j}\alpha_{ij}^{(2)}x_ix_j\,d\sigma \\ \frac{1}{2}\int_0^{2\pi}\left\{1+(2\pi-\sigma)\sum_{i,j}\frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j+\frac{1}{2!}\sum_{i,j}\left[(2\pi-\sigma)\frac{\sigma^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0)+(2\pi-\sigma)^2\left(\frac{\sigma\lambda(0)}{\partial\varepsilon_i}\right)\left(\frac{\sigma\lambda(0)}{\partial\varepsilon_i}\right)\right]\right. \\ \left.\varepsilon_i\varepsilon_j+0\left(\left|\varepsilon_i\varepsilon_j\right|\right)\right\}\times(\sin\sigma\sum\alpha_{ij}^{(1)}x_ix_j+\cos\sigma\sum\alpha_{\alpha ij}^{(2)}x_ix_j)\,d\sigma \\ \frac{1}{2}\int_0^{2\pi}\left\{I+(2\pi-\sigma)\frac{\partial D}{\partial\varepsilon_j}(0)\,\varepsilon_j+\frac{1}{2!}\left[(2\pi-\sigma)\frac{\partial^2 D}{\partial\varepsilon_i\,\partial\varepsilon_j}(0)+(2\pi-\sigma)^2\left(\frac{\partial\lambda(0)}{\partial\varepsilon_i}\right)\left(\frac{\partial\lambda(0)}{\partial\varepsilon_j}\right)\right]\right. \\ \left.\varepsilon_i\varepsilon_j+0\left(\left|\varepsilon_i\varepsilon_j\right|\right)\right\}\times e^{(2\pi-\sigma)D(0)}\times\begin{bmatrix}\sum_{i,j}\alpha_{ij}^{(3)}x_ix_j\\ \vdots\\ \sum_{i,j}\alpha_{ij}^{(n+m)}x_ix_j\end{bmatrix}d\sigma \end{bmatrix}$$

The above form of H (C,ε) shows that

$$\det\frac{\partial(H_1,\dots,H_{n+m})}{\partial(C_1,\dots,C_{n+m})}=0,$$

therefore the bifurcation equation (2.9) has non-unique solutions. But

$$\det\frac{\partial(H_3,\dots,H_{n+m})}{\partial(C_3,\dots,C_{n+m})}=\det(e^{2\pi D(0)}-I)\neq 0,$$

thus the implicit function Theorem 1.1 of [2] will allow us to solve $H_i(C,\varepsilon)=0$, $i=3,\dots,n+m$ uniquely. Consequently, we have $C_i=C_i(C_1,C_2,\varepsilon)$ such that for fixed

$$\varepsilon=0,\ C_3=C_4=\dots=C_{n+m}=0,$$

is an initial solution of $H_i(C,\varepsilon)=0$, $i=3,\dots,n+m$.

Putting $C_i(C_1,C_2,\varepsilon)$, $i=3,\dots,n+m$ into the first two equations of (2.9) yields

$$H_1(C_1,C_2,\varepsilon)=\left\{2\pi\sum\left[\frac{\partial\lambda}{\partial\varepsilon_j}(0)\right]\varepsilon_j+\frac{1}{2!}\sum\left[2\pi\frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0)+4\pi^2\left(\frac{\partial\lambda(0)}{\partial\varepsilon_i}\right)\left(\frac{\partial\lambda(0)}{\partial\varepsilon_j}\right)\right]\varepsilon_i\varepsilon_j+0\left(\left|\varepsilon_i\varepsilon_j\right|\right)\right\}C_1+$$

$$\frac{1}{2}\int_0^{2\pi}\left\{1+(2\pi-\sigma)\sum\frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j+\frac{1}{2!}\sum\left[(2\pi-\sigma)\times\frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0)+(2\pi-\sigma)^2\left(\frac{\partial\lambda(0)}{\sigma\varepsilon_i}\right)\frac{\partial\lambda(0)}{\partial\varepsilon_j}\right]\varepsilon_i\varepsilon_j+0\,(\varepsilon_i\varepsilon_j)\right\}\times$$

$$\left\{\cos\sigma \sum \alpha_{ij}^{(1)} x_i x_j - \sin\sigma \sum \alpha_{ij}^{(2)} x_i x_j \, d\sigma \right\} + 0|x^2| = 0, \qquad H_2(C_1, C_2, \varepsilon) = \left\{2\pi \sum \frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j + \frac{1}{2!}\sum\left[2\pi \frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0) + \right.\right.$$

$$\left.\left. 4\pi^2\left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_i}\right)\left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_j}\right)\right]\varepsilon_i\varepsilon_j + 0\left(|\varepsilon_i\varepsilon_j|\right)\right\} C_2 + \quad \frac{1}{2}\int_0^{2\pi}\left\{1 + (2\pi - \sigma)\sum\frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j + \frac{1}{2!}\sum\left[(2\pi - \sigma)\times\right.\right.$$

$$\left.\left.\frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0) + (2\pi - \sigma)^2\left(\frac{\partial\lambda\,(0)}{\sigma\varepsilon_i}\right)\left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_j}\right)\right]\varepsilon_i\varepsilon_j + 0\left(|\varepsilon_i\varepsilon_j|\right)\right\}\times \quad \left\{\sin\sigma\sum\alpha_{ij}^{(1)} x_i x_j + \cos\sigma\sum\alpha_{ij}^{(2)} x_i x_j \, d\sigma\right\} + 0(|x|)^2 = 0.$$

Recalling

$$x(\sigma, C, \varepsilon) = e^{\sigma B(\varepsilon)} + 0\left(|C|\right).$$

Thus

$$x(\sigma, C, 0) = \begin{bmatrix} x_1(\sigma, C, 0) \\ x_2(\sigma, C, 0) \\ \cdot \\ \cdot \\ \cdot \\ x_{n+m}(\sigma, C, 0) \end{bmatrix} = \begin{bmatrix} \cos\sigma & \sin\sigma & \\ -\sin\sigma & \cos\sigma & \\ & & e^{D(0)} \end{bmatrix} \begin{vmatrix} C_1 \\ C_2 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix} + 0\,|C| = \begin{bmatrix} C_1\cos\delta + C_2\sin\sigma \\ -C_1\sin\sigma + C_2\cos\sigma \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + 0\left(|C|\right)$$

and hence

$$H_1(C_1, C_2, \varepsilon) = P_1(C_1, \varepsilon) + P_2(C_1, C_2, \varepsilon) + P_3(C_1, C_2) + \text{hot} = 0$$

$$H_2(C_1, C_2, \varepsilon) = q_1(C_2, \varepsilon) + q_2(C_1, C_2, \varepsilon) + q_3(C_1, C_2) + \text{hot} = 0$$

where

$$P_1(C_1, \varepsilon) = \left\{2\pi\sum\frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j + \frac{1}{2!}\sum\left[2\pi\frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0) + \quad 4\pi^2\left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_i}\right)\left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_j}\right)\right]\varepsilon_i\varepsilon_j + 0\left(|\varepsilon_i\varepsilon_j|\right)\right\} C_1,$$

$$P_2(C_1, C_2, \varepsilon) = \frac{1}{2}\int_0^{2\pi}\left\{(2\pi - \sigma)\sum\frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j + \frac{1}{2!}\sum\left[(2\pi - \sigma)\frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0) + \right.\right.$$

$$\left.\left.(2\pi - \sigma)^2\left(\frac{\sigma\lambda\,(0)}{\partial\varepsilon_i}\right)\left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_j}\right)\right]\varepsilon_i\varepsilon_j + 0\left(|\varepsilon_i\varepsilon_j|\right)\right\}$$

$$\left\{(\cos\sigma)\left[\alpha_{11}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)^2 + 2\alpha_{12}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)(-C_1\sin\sigma + C_2\cos\sigma) + \right.\right.$$

$$\alpha_{22}^{(1)}(-C_1\sin\sigma + C_2\cos\sigma)^2\right] - (\sin\sigma)\left[\alpha_{11}^{(1)} \text{ unclear}$$

$$(C_1\cos\sigma + C_2\sin\sigma)^2 + 2\alpha_{12}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)$$

$$\left.\left.(-C_1\sin\sigma + C_2\cos\sigma) + \alpha_{22}^{(2)}(-C_1\sin\sigma + C_2\cos\sigma)^2\right]\right\} d\sigma\,\cdot$$

$$P_3(C_1, C_2) = \frac{1}{2}\int_0^{2\pi}\left\{(\cos\sigma)\left[\alpha_{11}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)^2 + \alpha_{12}^{(2)}(C_1\cos\sigma + C_2\sin\sigma)(-C_1\sin\sigma + C_2\cos\sigma)\right.\right.$$

$$+ \alpha_{22}^{(1)} (-C_1 \sin\sigma + C_2 \cos\sigma)^2] - (\sin\sigma) [\alpha_{11}^{(2)} (C_1 \cos\sigma + C_2 \sin\sigma)^2 + \alpha_{12}^{(2)} (C_1 \cos\sigma + C_2 \sin\sigma)(-C_1 \sin\sigma + C_2 \cos\sigma) + \alpha_{22}^{(2)} (-C_1 \sin\sigma + C_2 \cos\sigma)^2]\Big\} d\sigma$$

$$q_1 (C_2 \varepsilon) = \left\{ 2\pi \sum \frac{\partial\lambda}{\partial\varepsilon_i}(0)\varepsilon_j + \frac{1}{2!} \sum \left[2\pi \frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0) + 4\pi^2 \left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_i}\right)\left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_j}\right)\right] \varepsilon_i\varepsilon_j + 0(|\varepsilon_i\varepsilon_j|) \right\} C_2,$$

$$q_2 (C_1, C_2, \varepsilon) = \frac{1}{2}\int_0^{2\pi} \left\{ (2\pi - \sigma) \sum \frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j + \frac{1}{2!}\sum \left[(2\pi - \sigma) \times \frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0) + (2\pi - \sigma)^2 \left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_i}\right)\left(\frac{\partial\lambda\,(0)}{\partial\varepsilon_j}\right)\right] \varepsilon_i\varepsilon_j + 0(|\varepsilon_i\varepsilon_j|)\right\} \times \Big\{ (\sin\sigma)\big[\alpha_{11}^{(1)} (C_1 \cos\sigma + C_2 \sin\sigma)^2$$

$$+ 2\alpha_{12}^{(1)} (C_1 \cos\sigma + C_2 \sin\sigma)(-C_1 \sin\sigma + C_2 \cos\sigma) + \alpha_{22}^{(1)} (-C_1 \sin\sigma + C_2 \cos\sigma)^2\big] + (\cos\sigma) \times$$

$$\big[\alpha_{11}^{(2)} (C_1 \cos\sigma + C_2 \sin\sigma) + 2\alpha_{12}^{(2)} (C_1 \cos\sigma + C_2 \sin\sigma)\ (-C_1 \sin\sigma + C_2 \cos)\, \alpha_{22}^{(2)}\ (-C_1 \sin\sigma + C_2 \cos\sigma)^2\big]\Big\}\, d\sigma$$

$$q_3 (C_1, C_2) = \frac{1}{2}\int_0^{2\pi} \Big\{ \sin\sigma \big[\alpha_{11}^{(1)} (C_1 \cos\sigma + C_2 \sin\sigma)^2 + 2\alpha_{12}^{(1)} (C_1 \cos\sigma + C_2 \sin\sigma)(-C_1 \sin\sigma + C_2 \cos\sigma) +$$

$$\alpha_{22}^{(1)} (-C_1 \sin\sigma + C_2 \cos\sigma)^2\big] + (\cos\sigma) \times \big[\alpha_{11}^{(2)} (C_1 \cos\sigma + C_2 \sin\sigma)^2 + 2\alpha_{12} (C_1 \cos\sigma + C_2 \sin\sigma)$$

$$(-C_1 \sin\sigma + C_2 \cos\sigma) + \alpha_{22}^{(2)}\ (-C_1 \sin\sigma + C_2 \cos\sigma)^2\big]\Big\}\, d\sigma$$

and

$$\text{h. o. t} = 0\left(|\varepsilon_i\varepsilon_j| + |C_1|^2 + |C_2|^2 + |C_1C_2|\right).$$

Since $(\cos\sigma) \sum \alpha_{ij}^{(1)} x_i x_j - (\sin\sigma) \sum \alpha_{ij}^{(2)} x_i x_j$, $j=1,2$ are homogeneous degree 3 polynomials in $\sin\sigma$ and $\cos\sigma$. it is implied that

$$\int_0^{2\pi} \left[\cos\sigma \sum \alpha_{ij}^{(1)} x_i x_j - \sin\sigma \sum \alpha_{ij}^{(2)} x_i x_j \right] d\sigma = 0 \quad i,j=1,2,$$

hence, $P_3(C_1, C_2) = q_3 (C_1, C_2) = 0$ and H_1, H_2 become

$$H_1(C_1, C_2, \varepsilon) = P_1 (C_1, \varepsilon) + P_2 (C_1 C_2 \varepsilon) + \text{h. o. t} = 0,$$
$$H_2(C_1, C_2, \varepsilon) = q_1 (C_2, \varepsilon) + q_2 (C_1 C_2 \varepsilon) + \text{h. o. t} = 0.$$

Now three different cases are distinguished [3].

Case 1 $\sum_{j=1}^{m} \frac{\partial\lambda}{\partial\varepsilon_j}(0)\,\varepsilon_j = 0$, $\frac{\partial\lambda}{\partial\varepsilon_j}(0) \neq 0$ some $j=1, \dots, m$.

Then by assuming

$$\frac{1}{2!} \sum \frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0)\,\varepsilon_i\varepsilon_j = \gamma$$

we have

$$H_1(C_1, C_2, \gamma) = 2\pi\ \gamma C_1 + \gamma A_1 (C_1 C_2) + \text{h. o. t} = 0,$$

$$H_2(C_1,C_2,\gamma) = 2\pi\ \gamma C_2 + \gamma B_1(C_1, C_2) + \text{h. o. t} = 0, \tag{2.12}$$

where

$$A_1(C_1,C_2,) = \frac{1}{2}\int_0^{2\pi} (2\pi - \sigma)\Big\{\cos\sigma\Big[\alpha_{11}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)^2 + 2\alpha_{12}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)(-C_1\sin\sigma + C_2\cos\sigma$$

$$\alpha_{22}^{(1)}(-C_1\sin\sigma + C_2\cos\sigma)^2\Big] - \sin\sigma\Big[\alpha_{11}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)^2 + 2\alpha_{12}^{(2)}(C_1\cos\sigma + C_2\sin)(-C_1\sin\sigma + C_2\cos\sigma$$

$$+\alpha_{22}^{(2)}(-C_1\sin\sigma + C_2\cos\sigma)^2\Big]\Big\}d\sigma = \frac{\pi}{3}\Big\{(-2\alpha_{11}^{(1)} + 2\alpha_{22}^{(1)} + 10\alpha_{12}^{(2)})C_1C_2 +$$

$$\left(4\alpha_{11}^{2} - 2\alpha_{22}^{1} - \alpha_{22}^{2}\right)C_2 + \left(4\alpha_{22}^{2} + 2\alpha_{12}^{1} - \alpha_{11}^{2}\right)C_1^2\Big\},$$

$$B_1(C_1,C_2) = \frac{1}{2}\int_0^{2\pi}(2\pi - \sigma)\Big\{\sin\sigma\Big[\alpha_{11}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)^2 + 2\alpha_{12}^{(1)}(C_1\cos\sigma + C_2\sin\sigma)(-C_1\sin\sigma + C_2\cos\sigma) +$$

$$\alpha_{22}^{(1)}(-C_1\sin\sigma + C_2\cos\sigma)^2\Big] + \cos\sigma\Big[\alpha_{11}^{(2)}(C_1\cos\sigma + C_2\sin\sigma$$

$$+2\alpha_{12}^{(2)}(C_1\cos\sigma + C_2\sin\sigma)(-C_1\sin\sigma + C_2\cos\sigma) + \alpha_{22}^{(2)}(-C_1\sin\sigma + C_2\cos\sigma)^2\Big]\Big\}d\sigma$$

$$= \frac{\pi}{3}\Big\{\left(2\pi\ \alpha_{22}^{(2)} - 2\alpha_{11}^{(2)} - 10\alpha_{12}^{(1)}\right)C_1C_2 + \left(4\alpha_{11}^{(1)} - 2\alpha_{12}^{(2)} - \alpha_{22}^{(1)}\right)c_2^2 + \left(4\alpha_{22}^{(1)} - 2\alpha_{12}^{(2)} - \alpha_{11}^{(1)}\right)C_1^2\Big\},$$

and

$$\text{h. o. t.} = 0\left[\gamma + |C_1|^2 + |C_2|^2 + |C_1C_2|\right].$$

By dividing (3.2) by γ we obtain

$$K_1(C_1, C_2, \gamma) = 2\pi C_1 + A_1(C_1, C_2) + 0\left(|C_1|^2 + |C_2|^2 + |C_2C_2| + \gamma\right) = 0,$$

$$K_2(C_1, C_2, \gamma) = 2\pi C_2 + B_1(C_1, C_2) + 0\left(|C_1|^2 + |C_2|^2 + |C_1C_2| + \gamma\right) = 0. \tag{2.13}$$

To obtain zeros of (2.13) it is sufficient to obtain the point of intersections of $K_1 = 0$, $i = 1,2$.

The number of points of intersections depend heavily on

$0\left(|C_1|^2 + |C_2|^2 + |C_1C_2| + \gamma\right)$ and $\alpha_{ij}^{(k)}$, $i,j,k = 1,2$.

If the higher order terms are of degree 3, under appropriate values of $\alpha_{ij}^{(k)}$, there are at most none solutions and at least one solution.

Remark 2.1 In the special case, if $g(x,\varepsilon)$ be quadratic in x exactly, then the higher order terms with respect to C_1 and C_2 are $0(1)$. Hence the resulting equations are conic sections and there are at most four solutions subject to the values of $\alpha_{ij}^{(k)}$ and $K,i,j = 1,2$.

Case 2

$$\frac{\partial\lambda}{\partial\varepsilon}(0) = 0 \text{ for all } j = 1, \ldots, m.$$

The resultng bifurcation equations are the same as Case 1 and the analysis is the same.

Case 3

$$\frac{\partial\lambda}{\partial\varepsilon_j}(0) \neq 0 \text{ for some } j = 1, \ldots, m \text{ and}$$

$$\frac{\partial\lambda}{\partial\varepsilon_j}(0)\ \varepsilon_j \neq 0 \qquad \text{then by assuming}$$

$$\sum\frac{\partial\lambda}{\partial\varepsilon_j}(0)\ \varepsilon_j + \frac{1}{2!}\sum\left[\frac{\partial^2\lambda}{\partial\varepsilon_i\partial\varepsilon_j}(0)\right]\varepsilon_i\varepsilon_j = \gamma,$$

and

$$\frac{1}{2!}\sum\left[\left(\frac{\partial\lambda}{\partial\varepsilon_i}\right)\left(\frac{\partial\lambda}{\partial\varepsilon_j}\right)\right]\varepsilon_i\varepsilon_j = \eta,$$

we have the following bifurcation equations:

$H_1(C_1C_2, \gamma, \eta) = (2\pi\gamma + 4\pi^2\eta) C_1 + \gamma A_1(C_1, C_2) + \eta A_2(C_1, C_2) + \text{h.o.t.} = 0,$

$H_2(C_1C_2, \gamma, \eta) = (2\pi\gamma + 4\pi^2\psi) C_2 + \gamma B_1(C_1, C_2) + \eta B_2(C_1, C_2) + \text{h.o.t.} = 0.$

where

$$A_2(C_1C_2) = \frac{\pi}{3}\left\{\left(-2\alpha_{11}^{(1)} + 2\alpha_{22}^{(1)} + 10\alpha_{12}^{(2)}\right)C_1C_2 + \left(4\alpha_{11}^{(2)} - 2\alpha_{12}^{(1)} - \alpha_{22}^{(2)}\right)C_2^2 + \left(4\alpha_2^{(2)} + 2\alpha_{12}^{(1)} - \alpha_{11}^{(2)}\right)C_1^2\right\}$$

$$A_2(C_1C_2) = \left[\alpha_{11}^{(1)}\frac{4\pi^2}{3} - \alpha_{12}^{(1)}\frac{32\pi}{9} + \alpha_{22}^{(1)}\frac{4\pi^2}{3} + \alpha_{11}^{(2)}\frac{62\pi}{9} + \right.$$

$$\left.\alpha_{12}^{(2)}\frac{20\pi^2}{3} + \alpha_{22}^{(2)}\frac{62\pi}{9} + \frac{4\pi^2}{3}\left(-2\alpha_{11}^{(1)} + 2\alpha_{22}^{(1)} + 10\alpha_{12}^{(2)}\right\}\right]C_1C_2 +$$

$$\left[\alpha_{11}^{(1)}\frac{32\pi}{9} + \alpha_{12}^{(1)}\frac{4\pi^2}{3} + \alpha_{22}^{(1)}\frac{16\pi}{9} - \alpha_{11}^{(2)}\frac{8\pi^2}{3} + \alpha_{12}^{(2)}\frac{62\pi}{9} + \alpha_{22}^{(2)}\frac{2\pi^2}{3} + \frac{4\pi^2}{3}\left(4\alpha_{11}^{(1)} - 2\alpha_{12}^{(1)} - \alpha_{22}^{(2)}\right)\right]C_2^2 +$$

$$\left[\alpha_{11}^{(1)}\frac{16\pi}{9} - \alpha_{12}^{(1)}\frac{4\pi^2}{3} + \alpha_{22}^{(1)}\frac{32\pi}{9} - \alpha_{11}^{(2)}\frac{2\pi^2}{3} + \alpha_{12}^{(2)}\frac{62\pi}{9} - \alpha_{22}^{(2)}\frac{8\pi^2}{3} + \frac{4\pi^2}{3}\left(4\alpha_{22}^{(2)} - 2\alpha_{12}^{(1)} - \alpha_{11}^{(2)}\right)\right]C_1^2,$$

$$B_1(C_1C_2) = \frac{\pi}{3}\left\{\left(2\alpha_{22}^{(2)} - 2\alpha_{11}^{(2)} + 10\alpha_{12}^{(1)}\right)C_1C_2 + \left(4\alpha_{11}^{(1)} - 2\alpha_{12}^{(2)} - \alpha_{22}^{(1)}\right)C_2^2 + \left(4\alpha_{22}^{(1)} + 2\alpha_{12}^{(2)} - \alpha_{11}^{(1)}\right)C_1^2\right\}.$$

$$B_2(C_1C_2) = \left[\alpha_{11}^{(2)}\frac{4\pi^2}{3} - \alpha_{12}^{(2)}\frac{32\pi}{9} + \alpha_{22}^{(2)}\frac{4\pi^2}{3} + \alpha_{11}^{(1)}\frac{62\pi}{9} + \right.$$

$$\left.\alpha_{12}^{(1)}\frac{20\pi^2}{3} + \alpha_{22}^{(2)}\frac{62\pi}{9} + \frac{4\pi^2}{3}\left(-2\alpha_{11}^{(2)} + 2\alpha_{11}^{(2)} + 2\alpha_{22}^{(2)} - 10\alpha_{12}^{(2)}\right)\right]C_1C_2 +$$

$$\left[\alpha_{11}^{(2)}\frac{32\pi}{9} + \alpha_{12}^{(2)}\frac{4\pi^2}{3} + \alpha_{22}^{(2)}\frac{16\pi}{9} - \alpha_{11}^{(1)}\frac{8\pi^2}{3} + \alpha_{12}^{(1)}\frac{62\pi}{9} + \alpha_{22}^{(1)}\frac{2\pi^2}{3} + \frac{4\pi^2}{3}\left(4\alpha_{11}^{(1)} - 2\alpha_{12}^{(2)} - \alpha_{22}^{(1)}\right)\right]C_2^2 +$$

$$\left[\alpha_{11}^{(2)}\frac{16\pi}{9} - \alpha_{12}^{(2)}\frac{4\pi^2}{3} + \alpha_{22}^{(2)}\frac{32\pi}{9} - \alpha_{11}^{(1)}\frac{2\pi^2}{3} + \alpha_{12}^{(1)}\frac{62\pi}{9} - \alpha_{22}^{(1)}\frac{8\pi^2}{3} + \frac{4\pi^2}{3}\left(4\alpha_{22}^{(1)} - 2\alpha_{12}^{(2)} - \alpha_{11}^{(1)}\right)\right]C_1^2$$

and

$\text{h.o.t.} = 0\left(|C_1|^2 + |C_2|^2 + |C_1C_2| + \gamma + \eta\right).$

We divide both sides of (2.14) by η [3] and assuming

$\frac{2\pi\gamma}{\eta} + 4\pi^2 = \mu$ and

$\frac{\gamma}{\eta} = \upsilon$

To obtain

$\overline{H}_1(C_1, C_2, \mu, \upsilon) = \mu C_1 + \upsilon A_1(C_1, C_2) + A_2(C_1, C_2) + \text{h.o.t} = 0,$

$\overline{H}_2(C_1, C_2, \mu, \upsilon) = \mu C_2 + \upsilon B_1(C_1, C_2) + B_2(C_1, C_2) + \text{h.o.t} = 0,$

or equivalently

$\overline{H}(C, \mu, \upsilon) = \mu C + \upsilon Q_1(C) + Q_2(C) + \text{h.o.t} = 0,$

where

$$C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, Q_1(C)=\begin{bmatrix} A_1(C_1,C_2) \\ B_1(C_1,C_2) \end{bmatrix}, Q_2(C)=\begin{bmatrix} A_2(C_1,C_2) \\ B_2(C_1,C_2) \end{bmatrix}$$

and

$$\text{h.o.t} = 0\left(|C_1|^2 + |C_2|^2 + |C_1C_2| + \upsilon\right).$$

Our bifurcation equations consist of studying the simultaneous solutions of $\overline{H}_1=0, \overline{H}_2=0,$ which by following Hale [3] and Mallet-Paret [4] it is easily done.

References

1. Cronin, J. Bifurcation of periodic solution, J. Math. Anal. and Applications, **68**, 130-151, (1979).
2. Vainberg, M.N. and Trenogin, V.A. Theory of Branching Solutions of Non-linear Equation, Noordhoff, Chapter 1 (1974).
3. Hale, J.K., Bifurcation in two dimensions quadratic non-linearities and two parameters, Non-Linear Analysis and Mechanics. Herriot-Watt Symposium, Vol. 1, No. 17, (1977).
4. Mallet-Paret, J., Buckling of cylindrical shells with small curvature. Quarterly of Applied Mathematics, 383-400, (October, 1977).